Irreducibility of SPDEs driven by pure jump noise

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Outline

Introduction

Let H be a topological space with Borel σ-field $\mathcal{B}(H)$, and let $\mathbb{X} := \{X^x(t), t \geq 0; x \in H\}$ be an H-valued Markov process on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. $\mathbb{X}$ is said to be irreducible in H if for each $t > 0$ and $x \in H$

$$\mathbb{P}(X^x(t) \in B) > 0 \quad \text{for any non-empty open set } B.$$

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  - Irreducibility and the strong Feller property:
    S. Peszat, J. Zabczyk, Ann. Probab. (1995)
  - Irreducibility and the asymptotic strong Feller property:
    M. Hairer, J.C. Mattingly, Ann. of Math. (2006)
  - Irreducibility and the *e*-property:
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- The recurrence of Markov processes:
  G. Da Prato and J. Zabczyk,(1996) Ergodicity for Infinite-Dimensional Systems

Introduction

The study of the irreducibility for stochastic dynamical systems driven by Gaussian noise has a long history:

- J. L. Doob, Asymptotic property of markov transtion probability. Trans. Amer. Math. Soc., 64 (1948) 393-421
- S. Peszat, J. Zabczyk, Strong Feller property and irreducibility for diffusions on Hilbert spaces. Ann. Probab. 1995, 157-172.
- G. Da Prato and J. Zabczyk, Ergodicity for Infinite-Dimensional Systems, London Math.Soc. Lecture Note Ser. 229, Cambridge University Press, Cambridge, UK, 1996
- G. Da Prato, Kolmogorov equations for stochastic PDEs. Advanced Courses in Mathematics. CRM Barcelona. Birkhauser Verlag, Basel, 2004. x+182 pp.

To obtain the irreducibility for stochastic equations driven by Gaussian noise, one usually needs to solve a control problem. In doing so, three ingredients play very important role:

- the (approximate) controllability of the associated PDEs,
- Girsanov's transformation of Wiener processes,
- the support of Wiener processes/stochastic convolutions on path spaces.

Introduction

However, things become quite different when the driving noises are pure jump processes. It seems that no results on the irreducibility of stochastic dynamical systems driven by pure jump noise had been available before the paper E. Priola, J. Zabczyk, PTRF (2011).

Compared with the case of the Gaussian driving noise, there are very few results on the irreducibility of the case of the pure jump driving noise, because the systems behave drastically differently due to the appearance of jumps.

Introduction

Now we mention the results on the irreducibility of SPDEs driven by pure jump noise. To do this, we first introduce the so-called cylindrical pure jump Lévy processes defined by the orthogonal expansion

$$L(t) = \sum_i \beta_i L_i(t) e_i, \quad t \geq 0, \tag{2.1}$$

where $\{e_i\}$ is an orthonormal basis of a separable Hilbert space H, $\{L_i\}$ are real valued i.i.d. pure jump Lévy processes, and $\{\beta_i\}$ is a given sequence of non zero real numbers.

Introduction (SPDEs)

E. Priola, J. Zabczyk, Structural properties of semilinear SPDEs driven by cylindrical stable process, Probab. Theory Relat. Fields 149 (2011), 97-137.

Noise: cylindrical symmetric α-stable processes, $\alpha \in (0, 2)$, which have the form (2.1) with $\{L_i\}$ replaced by real valued i.i.d. symmetric α-stable processes.

Introduction (SPDEs)

R. Wang, J. Xiong, L. Xu, Irreducibility of stochastic real Ginzburg-Landau equation driven by α-stable noises and applications. Bernoulli, 23 (2) (2017), 1179-1201.

Noise: cylindrical symmetric α-stable processes with $\alpha \in (1, 2)$.

Some technical restrictions are placed on the driving noises. For example,

$$\alpha \in (1, 2) \tag{2.2}$$

and

$$C_1 \gamma_i^{-\beta} \leq |\beta_i| \leq C_2 \gamma_i^{-\beta} \tag{2.3}$$

with $\beta > \frac{1}{2} + \frac{1}{2\alpha}$ for some positive constants C_1 and C_2. Here $\{\gamma_i = 4\pi^2 |i|^2\}$ are the eigenvalues of the Laplace operator on H.

Introduction (SPDEs)

- R. Wang, L. Xu, Asymptotics for stochastic reaction-diffusion equation driven by subordinate Brownian motion. Stochastic Process. Appl. 128 (5) 2018, 1772-1796.
- Z. Dong, F. Wang, L. Xu, Irreducibility and asymptotics of stochastic Burgers equation driven by α-stable processes. Potential Anal. 52 (2020), no. 3, 371-392.

Noise: the subordinated cylindrical Wiener process with a $\alpha/2$-stable subordinator, $\alpha \in (1, 2)$.

Technical restrictions as (2.2) and (2.3) on the driving noises are required.

[FHR] P. Fernando, E. Hausenblas, P. Razafimandimby, Irreducibility and exponential mixing of some stochastic hydrodynamical systems driven by pure jump noise. Comm. Math. Phys. 348 (2016), no. 2, 535-565.

Noise: the driving noises L they considered are of the form (2.1) satisfying

(A1) The intensity measure μ of each component process L_i satisfies that there exists a strictly monotone and C^1 function $q : (0,\infty) \to (0,\infty)$ such that

$$\lim_{r \nearrow \infty} q(r) = 0, \lim_{r \searrow 0} q(r) = 1, \mu(dz) = q(|z|)|z|^{-1-\theta} dz, \theta \in (0,2);$$

$$\int_{\mathbb{R}} (1 - q^{1/2}(|z|))^2 \mu(dz) < \infty. \tag{2.4}$$

(A2) There exist a certain $\epsilon \in (0,2)$ and $\vartheta \in [0,1/2)$ such that

$$\sum_i (|\beta_i| + \beta_i^2 \lambda_i^{-2\vartheta} + \beta_i^2 \lambda_i^{\epsilon-1} + \beta_i^4 \lambda_i^{\epsilon}) < \infty. \quad (2.5)$$

Here $0 < \lambda_1 < \lambda_2 < \ldots$ are the eigenvalues associated with a positive self-adjoint operator appearing in the equations they studied.

Open problem: The driving noises L could not cover cylindrical α-stable processes. As mentioned by the authors, their results could not be applied to the family of truncated Lévy flights which requires that q appearing in (A1) satisfies $q(r) \equiv 0, r \notin (0,R]$ for some $R > 0$. The authors also pointed out that the irreducibility in their framework with the driving noises replaced by stable processes does not follow from their results and is still an open problem.

- The existing methods are basically along the same lines as that of the Gaussian case.
- The methods and techniques available for dynamical systems driven by Gaussian noise are not well suited for investigating the irreducibility of systems driven by jump type noise for two main reasons.

Introduction

- One is that there exist very few results on the support of the pure jump Lévy processes/the stochastic convolutions on path spaces. Due to the discontinuity of trajectories, the characterization of the support of the pure jump processes is much harder than that of the Gaussian case. They heavily rely on the fact that the driving noises are additive type and more or less in the class of stable processes.

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- The other is that Girsanov's transformation of the pure jump Lévy process is much less effective than that of Gaussian case, because the density of the Girsanov transform of a Poisson random measure is expressed in terms of nonlinear invertible and predictable transformations, and is to censore jumps or thin the size of jumps.

The use of the existing methods to deal with the case of other types of additive pure jump noises appears to be unclear, let alone the case of multiplicative noises.

So far, there is a lack of effective methods to obtain the irreducibility of stochastic equations driven by pure jump noise.

This strongly motivates our study.

Introduction (SPDEs)

Summary: Although, the irreducibility of several interesting SPDEs driven by pure jump noise is obtained, there are always some very restrictive assumptions on the driving Lévy noise: The driving noises are additive type and more or less in the class of stable processes, and other technical assumptions such as (2.2)-(2.5) are required.

Actually, the conditions on the driving noise to obtain the irreducibility are much stronger than that to obtain the well posedness.

Introduction (SDEs)

A. Arapostathis, G. Pang, N. Sandric, Ergodicity of a Lévy-driven SDE arising from multiclass many-server queues. Ann. Appl. Probab. 29 (2019), no. 2, 1070–1126.

They obtained the irreducibility of a class of multidimensional Ornstein-Uhlenbeck processes driven by additive pure jump noise L.

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They obtained the irreducibility of a class of multidimensional Ornstein-Uhlenbeck processes driven by additive pure jump noise L.

L is of the form $L(t) = L_1(t) + L_2(t)$, where L_1 and L_2 are independent d-dimensional pure jump Lévy processes, such that one of the following conditions is satisfied

- (1) L_1 is a subordinate Brownian motion, and L_2 can be any pure jump Lévy process or vanish;
- (2) L_1 is an anisotropic Lévy process with independent symmetric one dimensional α-stable components for $\alpha \in (0, 2)$, and L_2 is a compound Poisson process.

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Their proofs are based on studying the corresponding infinitesimal generator of the Ornstein-Uhlenbeck processes.

Introduction (SDEs)

L. Xie, X. Zhang, Ergodicity of stochastic differential equations with jumps and singular coefficients. Ann. Inst. Henri Poincaré Probab. Stat., 56(1) 2020, 175-229.

Noise: symmetric and rotationally invariant α-stable processes with $\alpha \in (1, 2)$.

Our results: we establish the irreducibility for a class of SDE with singular coefficients driven by non-degenerate α-stable-like Lévy process with $\alpha \in (0, 2)$. We notice that the study of the supercritical case $\alpha \in (0, 1)$ is much harder and attracts a lot of attention. Our results for the supercritical case are new. We stress that our results for the case of the additive noise are sharp. See Proposition 5.5.

Z. Chen, X. Zhang, G. Zhao, Supercritical SDEs driven by multiplicative stable-like Lévy processes. Trans. Amer. Math. Soc. 374 (2021), no. 11, 7621-7655.

Let $(Z, \mathcal{B}(Z))$ be a metric space, and ν a given σ-finite measure ν on it, that is, there exists $Z_n \in \mathcal{B}(Z), n \in \mathbb{N}$ such that $Z_n \uparrow Z$ and $\nu(Z_n) < \infty, \forall n \in \mathbb{N}$. Let $N : \mathcal{B}(Z \times \mathbb{R}^+) \times \Omega \to \bar{\mathbb{N}} := \mathbb{N} \cup \{0, \infty\}$ be a time homogeneous Poisson random measure on $(Z, \mathcal{B}(Z))$ with intensity measure ν.

Now we consider the following SPDEs driven by pure jump noise:

$$\begin{aligned} dX(t) &= \mathcal{A}(X(t))dt + \int_{Z_1^c} \sigma(X(t-), z)\tilde{N}(dz, dt) + \int_{Z_1} \sigma(X(t-), z)N(dz, dt), \\ X(0) &= x, \end{aligned} \tag{3.1}$$

where $\mathcal{A} : V \to V^*$ and $\sigma : H \times Z \to H$ are Borel measurable mappings, and, for any $m \in \mathbb{N}$, Z_m^c denotes the complement of Z_m relative to Z.

Hypotheses

Define $\mathbb{F}$-stopping times

$$\tau_{x,y}^{\eta} = \inf\{t \geq 0 : X^x(t) \notin B(y, \eta)\}. \tag{3.2}$$

Assumption 3.1

For any $x \in H$, there exists a unique global solution $X^x = \left(X^x(t)\right)_{t\geq 0}$ to (3.1) and $\{X^x, x \in H\}$ forms a strong Markov process.

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Assumption 3.2

For any $h \in H$, there exists $\eta_h > 0$ such that, for any $\eta \in (0, \eta_h]$, there exist $(\epsilon, t) = (\epsilon(h,\eta), t(h,\eta)) \in (0, \frac{\eta}{2}] \times (0, \infty)$ satisfying,

$$\inf_{\tilde{h} \in B(h,\epsilon)} \mathbb{P}(\tau^{\eta}_{\tilde{h},h} \geq t) > 0.$$

$$\inf_{\tilde{h} \in B(h,\epsilon)} \mathbb{P}(\sup_{s \in [0,t]} ||X^{\tilde{h}}(s) - h||_H \leq \eta) > 0.$$

Hypotheses

Assumption 3.3

For any $\hbar, y \in H$ *with* $\hbar \neq y$ *and any* $\bar{\eta} > 0$*, there exist* $n, m \in \mathbb{N}$*, and* $\{l_i, i = 1, 2, ..., n\} \subset Z_m$ *such that, for any* $\eta \in (0, \frac{\bar{\eta}}{2})$*, there exist* $\{\epsilon_i, i = 1, 2, ..., n\} \subset (0, \infty)$ *and* $\{\eta_i, i = 0, 1, ..., n\} \subset (0, \infty)$ *such that, denoting*

$$q_0 = \hbar, \ q_i = q_{i-1} + \sigma(q_{i-1}, l_i), \ i = 1, 2, ..., n,$$

- $0 < \eta_0 \leq \eta_1 \leq ... \leq \eta_{n-1} \leq \eta_n \leq \eta$;
- *for any* $i = 0, 1, ..., n-1$,
  $\{\tilde{q} + \sigma(\tilde{q}, l), \ \tilde{q} \in B(q_i, \eta_i), l \in B(l_{i+1}, \epsilon_{i+1})\} \subset B(q_{i+1}, \eta_{i+1})$;
- $B(q_n, \eta_n) \subset B(y, \frac{\bar{\eta}}{2})$;
- *for any* $i = 1, 2, ..., n$, $\nu(B(l_i, \epsilon_i)) > 0$;
- *there exists* $m_0 \geq m$ *such that* $\bigcup_{i=1}^{n} B(l_i, \epsilon_i) \subset Z_{m_0}$.

Main results

Theorem 3.1

Suppose Assumptions 3.1, 3.2 and 3.3 hold. Then the Markov process formed by the solution $\{X^x, x \in H\}$ of equation (3.1) is irreducible in H.

Hypotheses on additive noise

Now consider the SPDE (3.1) with the additive noise $dL(t)$, that is,

$$\begin{aligned} dX(t) &= \mathcal{A}(X(t))dt + dL(t), \\ X(0) &= x. \end{aligned} \tag{3.3}$$

Let us now formulate the condition on the jumping measure in this setting.

Assumption 3.4

For any $h \in H$ and $\eta_h > 0$, there exist $n \in \mathbb{N}$, a sequence of strict positive numbers $\eta_1,\ \eta_2, \cdots, \eta_n$, and $a_1,\ a_2, \cdots,\ a_n \in H \setminus \{0\}$, such that $0 \notin \overline{B(a_i, \eta_i)}$, $\nu\big(B(a_i, \eta_i)\big) > 0$, $i = 1, ..., n$, and that $\sum_{i=1}^n B(a_i, \eta_i) := \{\sum_{i=1}^n h_i : h_i \in B(a_i, \eta_i), 1 \le i \le n\} \subset B(h, \eta_h)$.

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For any measure ρ on H, its support $S_\rho = S(\rho)$ is defined to be the set of $x \in H$ such that $\rho(G) > 0$ for any open set G containing x. Set

$$H_0 := \Big\{ \sum_{i=1}^{n} m_i a_i, \ \ n, m_1, ..., m_n \in \mathbb{N}, \ \ a_i \in S_\nu \Big\}. \tag{3.4}$$

It is not difficult to see that Assumption 3.4 holds if and only if H_0 is dense in H.

Main results

As an application of Theorem 3.1, we have

Theorem 3.2

Under Assumptions 3.1, 3.2 and 3.4, the Markov process formed by the solution $\{X^x, x \in H\}$ of equation (3.3) is irreducible in H.

Considering (3.3) with $\mathcal{A} = 0$ and $x = 0$, then we have

Corollary 1

Assume that Assumptions 3.4 holds. For any $s > 0$, $\epsilon > 0$ and $h \in H$, $\mathbb{P}(L(s) \in B(h, \epsilon)) > 0$.

Remark 3.1

The result in Theorem 3.2 is optimal in the sense that it is false when Assumption 3.4 fails. Here is an example. Consider SDE on $\mathbb{R}$*:*

$$dX(t) = b(X(t))dt + dL(t), \quad X(0) = x \in \mathbb{R}. \tag{3.5}$$

- $b : \mathbb{R} \to \mathbb{R}^+$ *is Borel measurable*
- $L = \{L(t) = \int_0^t \int_{z>0} zN(dz, ds), t \geq 0\}$ *is a Lévy process on* $\mathbb{R}$*,* *the intensity measure* ν*:* $\nu(\{0\}) = 0$*,* $\nu(\{z < 0\}) = 0$ *and* $\int_{z>0} z\nu(dz) < \infty$*.*

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In this case, Assumption 3.4 obviously fails. It is easy to see that any solution X^x *to (3.5) (if it exists) satisfies*

$$\mathbb{P}(X^x(t) \in (-\infty, x)) = 0, \ \forall t \geq 0,$$

which in particular means that $\{X^x, x \in \mathbb{R}\}$ *is not irreducible in* $\mathbb{R}$*.*

Idea of the Proof

For any $x, y \in H$, $T > 0$ and $\kappa > 0$, our aim is to prove that

$$\mathbb{P}\Big(X^x(T) \in B(y, \kappa)\Big) > 0. \tag{4.1}$$

Here, for any $h \in H$ and $l > 0$, denote $B(h, l) = \{\hbar \in H : ||\hbar - h||_H < l\}$.

- Assumption 3.2: a weakly continuous assumption on $\mathbb{X}$ uniformly in the initial data.
- Assumption 3.3: a nondegenerate condition on the intensity measure of the driving Lévy noise, which basically says that for any $\hbar, \overrightarrow{\hbar} \in H$, the neighbourhoods of $\overrightarrow{\hbar}$ can be reached with positive probability from $\hbar$ through a finite number of choosing jumps.

Applying Assumption 3.2 to the given y and κ, there exist $\epsilon_0 := \epsilon(y, \frac{\kappa}{2}) \in (0, \frac{\kappa}{4})$ and $t_0 := t(y, \frac{\kappa}{2}) > 0$ such that for any $\hbar \in B(y, \epsilon_0)$,

$$\mathbb{P}\Big(\{X^{\hbar}(t) \in B(y, \frac{\kappa}{2}), \forall t \in [0, t_0]\}\Big) > 0. \tag{4.2}$$

Therefore, set $T_0 = T - \frac{t_0}{2}$, once we prove that there exists $\widetilde{T} \in (T_0, T)$ such that

$$\mathbb{P}(X^x(\widetilde{T}) \in B(y, \epsilon_0)) > 0, \tag{4.3}$$

by the Markov property of $\mathbb{X}$, (4.1) follow from (4.2) and (4.3), completing the proof.

We now explain the ideas of proving (4.3). First, notice that there exists $\zeta \in H$ such that for any $\rho > 0$

$$\mathbb{P}\big(X^x(T_0) \in B(\zeta,\rho)\big) > 0. \tag{4.4}$$

By Assumption 3.3, $B(y,\epsilon_0)$ can be reached with positive probability from ζ through a finite number of choosing jumps. Set σ_i be the i-th jump time. One key step to obtain (4.3) is to prove that there exist $\rho_0 > 0$, $\rho_1 > 0$, $q_1 \in H$, and $T_1 \in (T_0, T)$ such that

$$\mathbb{P}\Big(\{X^x(T_0) \in B(\zeta,\rho_0)\} \cap \{X^x(t) \in B(\zeta,2\rho_0), \forall t \in [T_0,\sigma_1)\} \\ \cap\{X^x(\sigma_1) \in B(q_1,\frac{\rho_1}{2})\} \cap \{X^x(t) \in B(q_1,\rho_1), \forall t \in (\sigma_1, T_1]\}\Big) > 0,$$

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which implies that

$$\mathbb{P}\Big(\{X^x(T_0) \in B(\zeta, \rho_0)\} \cap \{X^x(T_1) \in B(q_1, \rho_1)\}\Big) > 0. \tag{4.5}$$

To complete the proof of (4.5), a further delicate argument is carried out, which requires an intricate cutoff procedure and employs stopping time techniques, etc.

The argument exploits the strong Markov property of $\mathbb{X}$, the fact that the jumps of the Poisson random measure on disjoint subsets are mutually independent, and the fact that with probability one, two independent Lévy processes can not jump simultaneously at any given moment, etc.

It also relies on carefully choosing moments and sizes of jumps of the driving noises.

Following a recursive procedure we are able to prove that there exist $\{q_i, i = 1, 2, ..., n\} \subseteq H$, $\{\rho_i, i = 1, 2, ..., n\} \subseteq (0, \infty)$ and $T_0 < T_1 < T_2 < ... < T_n < T$ such that

$$\mathbb{P}\Big(\{X^x(T_0) \in B(\zeta, \rho_0)\} \cap_{i=1}^n \{X^x(T_i) \in B(q_i, \rho_i)\}\Big) > 0. \tag{4.6}$$

Carefully choosing $B(q_n, \rho_n) \subset B(y, \epsilon_0)$, the above inequality implies that

$$\mathbb{P}\Big(\{X^x(T_n) \in B(y, \epsilon_0)\}\Big) \geq \mathbb{P}\Big(\{X^x(T_n) \in B(q_n, \rho_n)\}\Big) > 0.$$

Therefore, (4.3) holds, completing the proof.

Applications and examples

For any measure ρ on H, its support $S_\rho = S(\rho)$ is defined to be the set of $x \in H$ such that $\rho(G) > 0$ for any open set G containing x. Set

$$H_0 := \Big\{ \sum_{i=1}^{n} m_i a_i, \ \ n, m_1, ..., m_n \in \mathbb{N}, \ \ a_i \in S_\nu \Big\}. \tag{5.1}$$

It is not difficult to see that

$$\text{Assumption 3.4 holds} \Leftrightarrow H_0 \text{ is dense in } H.$$

Examples for Assumption 3.4

Example 5.1

Let $H = \mathbb{R}$*. The intensity measure* ν *of the Lévy process satisfies Assumption 3.4, namely,* H_0 *defined in (5.1) is dense in* $\mathbb{R}$*, if one of the following conditions is satisfied:*

(1) *There exist* $a < 0$*,* $b > 0$ *and* $c_n \neq 0, n \in \mathbb{N}$*, such that* $\lim_{n\to\infty} c_n = 0$*,* $\{a, b, c_n, n \in \mathbb{N}\} \subseteq S_\nu$*.*

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(2) $\nu(\mathbb{R}) = \infty$*, and there exist* $a > 0$ *and* $b < 0$ *such that* $\{a, b\} \subseteq S_\nu$*.*

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Let $H = \mathbb{R}$. The intensity measure ν of the Lévy process satisfies Assumption 3.4, namely, H_0 defined in (5.1) is dense in $\mathbb{R}$, if one of the following conditions is satisfied:

(1) *There exist $a < 0$, $b > 0$ and $c_n \neq 0, n \in \mathbb{N}$, such that $\lim_{n\to\infty} c_n = 0$, $\{a, b, c_n, n \in \mathbb{N}\} \subseteq S_\nu$.*

(2) *$\nu(\mathbb{R}) = \infty$, and there exist $a > 0$ and $b < 0$ such that $\{a, b\} \subseteq S_\nu$.*

(3) *There exist $a \neq 0$ and $b_n \neq -a, n \in \mathbb{N}$, such that $\lim_{n\to\infty} b_n = -a$, $\{a, b_n, n \in \mathbb{N}\} \subseteq S_\nu$.*

(4) *There exist $a \neq 0$ and $b_n \neq 0, n \in \mathbb{N}$, such that $\lim_{n\to\infty} b_n = \infty$ and there exists a subsequence of $\{na + b_n, n \in \mathbb{N}\}$ strictly increase (or decrease) to 0, $\{a, b_n, n \in \mathbb{N}\} \subseteq S_\nu$.*

(5) *Set $S_\nu^+ = \{a \in S_\nu : a > 0\}$ and $S_\nu^- = \{a \in S_\nu : a < 0\}$. $Leb(S_\nu^+) > 0$ and $Leb(S_\nu^-) > 0$. Here Leb is the Lebesgue measure on $\mathbb{R}$.*

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Let $H = \mathbb{R}$. The intensity measure ν of the Lévy process satisfies Assumption 3.4, namely, H_0 defined in (5.1) is dense in $\mathbb{R}$, if one of the following conditions is satisfied:

(1) *There exist $a < 0$, $b > 0$ and $c_n \neq 0, n \in \mathbb{N}$, such that $\lim_{n\to\infty} c_n = 0$, $\{a, b, c_n, n \in \mathbb{N}\} \subseteq S_\nu$.*

(2) *$\nu(\mathbb{R}) = \infty$, and there exist $a > 0$ and $b < 0$ such that $\{a, b\} \subseteq S_\nu$.*

(3) *There exist $a \neq 0$ and $b_n \neq -a, n \in \mathbb{N}$, such that $\lim_{n\to\infty} b_n = -a$, $\{a, b_n, n \in \mathbb{N}\} \subseteq S_\nu$.*

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(5) *Set $S_\nu^+ = \{a \in S_\nu : a > 0\}$ and $S_\nu^- = \{a \in S_\nu : a < 0\}$. $Leb(S_\nu^+) > 0$ and $Leb(S_\nu^-) > 0$. Here Leb is the Lebesgue measure on $\mathbb{R}$.*

(6) *Let S_ν^+ and S_ν^- defined as in (5). There exist $a \in S_\nu^+$ and $b \in S_\nu^-$ such that a/b is an irrational number.*

Remark 5.1

The so-called tempered stable processes on $\mathbb{R}$ *have the intensity measure* ν *given by*

$$\nu(dx) = q^+(x)1_{(0,\infty)}(x)\frac{1}{x^{1+\beta^+}}dx + q^-(-x)1_{(-\infty,0)}(x)\frac{1}{|x|^{1+\beta^-}}dx, \quad (5.2)$$

here $q^+ : (0,\infty) \to [0,\infty)$ *and* $q^- : (0,\infty) \to [0,\infty)$ *are called tempering functions,* β^+ *and* β^- *are positive constants. One can easily see that under mild conditions on* q^+ *and* q^-*, the intensity measure* ν *satisfies one of the conditions in Example 5.1.*

Examples for Assumption 3.4

Example 5.2

Let $H = \mathbb{R}^d$, $d \in \mathbb{N} \cup \{+\infty\}$. *Let* $\{e_1, e_2, ..., e_d\}$ *be an orthonormal basis of* H. *Let* $\{L_i(t), t \geq 0\}_{i \in \mathbb{N}}$ *be mutually independent one dimensional pure jump Lévy processes with their intensity measures* ν_i *satisfying one of the conditions listed in Example 5.1. Choosing* $\beta_i \in \mathbb{R} \setminus \{0\}, i \in \mathbb{N}$ *such that*

$$\int_H ||h||_H^2 \wedge 1 \nu(dh) = \sum_{i=1}^{d} \int_{\mathbb{R}} |\beta_i x_i|^2 \wedge 1 \nu_i(dx_i) < \infty. \tag{5.3}$$

$L(t) = \sum_{i=1}^{d} \beta_i L_i(t) e_i, t \geq 0$ *defines an* H*-valued Lévy process, and its intensity measure* ν *satisfies Assumption 3.4.*

Examples for Assumption 3.4

Example 5.3 (Subordination of Lévy processes)

Now let $\{Z_t, t \geq 0\}$ *be a subordinator with Lévy measure* ρ *satisfying* $\int_{(0,\infty)}(1 \wedge s)\rho(ds) > 0$*. We have the following two concrete examples.*

- *(1) Let* $\{X_t, t \geq 0\} = \{W_t, t \geq 0\}$ *be a Q-Wiener process on* H*,* $Q \in L(H)$ *is nonnegative, symmetric, with finite trace and* $KerQ = \{0\}$*; here* $L(H)$ *denotes the set of all bounded linear operators on* H*.* $L_t = W_{Z_t}, t \geq 0$*.*
- *(2) Let* $\{X_t, t \geq 0\}$ *be a Lévy process introduced in Example 5.2.* $L_t = X_{Z_t}, t \geq 0$*.*

Examples for Assumption 3.4

Example 5.4

Let $H = \mathbb{R}^d, d \in \mathbb{N}$. Assume that the intensity measure ν of the Lévy process is absolutely continuous with respect to the Lebesgue measure dz on $\mathbb{R}^d$, that is, $\nu(dz) = q(z)dz$, for some measurable function $q : \mathbb{R}^d \to [0, \infty)$. Let $\{e_1, e_2, ..., e_d\}$ be an orthonormal basis of $\mathbb{R}^d$. Assume that q is a continuous function and that $q(x) > 0$ for $x = e_i, i = 1, 2, ..., d$ and $x = -\sum_{j=1}^d e_i$. Then we can easily see that the intensity measure ν satisfies Assumption 3.4.

One can find other mild conditions on q such that the intensity measure ν satisfies Assumption 3.4, even for the case that q is not a continuous function.

Remark 5.2

Let $H = \mathbb{R}^d, d \in \mathbb{N} \cup \{+\infty\}$*. The driving Lévy processes on* H *satisfying Assumption 3.4 include not only a large class of compound Poisson processes, but also a large class of heavy-tailed Lévy processes, even the processes who's intensity measures* ν *satisfy for any small* $\alpha > 0$*,* $\int_{||z||_H > 1} ||z||_H^\alpha \nu(dz) = \infty$*.*

Examples for Assumption 3.3

Example 5.5

Let $H = \mathbb{R}^d, d \in \mathbb{N} \cup \{+\infty\}$. *Assume that the noise term in (3.1) has the form:*

$$\begin{aligned} & \int_{Z_1^c} \sigma(X(t-), z)\tilde{N}(dz, dt) + \int_{Z_1} \sigma(X(t-), z)N(dz, dt) \\ = \ & \int_{Z_1^c} \sigma_1(X(t-), z)\tilde{N}_1(dz, dt) + \int_{Z_1} \sigma_1(X(t-), z)N_1(dz, dt) + dL(t), \end{aligned}$$

here L *is a Lévy process on* H *satisfying Assumption 3.4,* N_1 *can be any Poisson random measure on* Z*, and* L *and* N_1 *are independent. Then Assumption 3.3 holds.*

Examples for Assumption 3.3

Example 5.6

Let $H = \mathbb{R}^d, d \in \mathbb{N} \cup \{+\infty\}$. Recall $L = \{L_t = W_{Z_t}, t \geq 0\}$ in Example 5.3, where $W = \{W_t, t \geq 0\}$ is a Q-Wiener process on H, $Q \in L(H)$ is nonnegative, symmetric, with finite trace and $KerQ = \{0\}$, $Z = \{Z_t, t \geq 0\}$ is a subordinator with Lévy measure ρ satisfying $\int_{(0,\infty)}(1 \wedge s)\rho(ds) > 0$, and W and Z are independent. The intensity measure of L is denoted by ν, which satisfies $S_\nu = H$. Denote by N the Poisson random measure corresponding to L, and $\tilde{N}$ the associated compensated Poisson random measure.

Example 5.6 continued

Denote by $L_2(Q^{1/2}(H), H)$ the space of all Hilbert-Schmidt operators from $Q^{1/2}(H)$ to H equipped with the Hilbert-Schmidt norm. Assume that $\sigma : H \to L_2(Q^{1/2}(H), H)$ is continuous, and, for any $h \in H$, $Ker\sigma(h) = 0$. Then the driving noise

$$\begin{aligned} & \int_0^t \sigma(X(s-))dL_s \\ = & \int_0^t \int_{0<||z||_H \leq 1} \sigma(X(s-))zd\tilde{N}(dz, ds) + \int_0^t \int_{||z||_H > 1} \sigma(X(s-))zdN(dz, ds), t \geq 0 \end{aligned}$$

satisfies Assumption 3.3.

Example 5.7

Let $H = \mathbb{R}^d, d \in \mathbb{N} \cup \{+\infty\}$, *and* $\{e_i, i = 1, 2, ..., d\}$ *be an orthonormal basis of* H. *Let* $\{L_i = \{L_i(t), t \geq 0\}, i \in \mathbb{N}\}$ *be a sequence of i.i.d. one dimensional Lévy processes with intensity measure* μ *on some filtered probability space* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. *Assume that*

(C1) *there exists* $c_n^+ > 0, c_n^- < 0, n \in \mathbb{N}$ *such that* $\lim_{n\to\infty} c_n^- = \lim_{n\to\infty} c_n^+ = 0$ *and* $\{c_n^+, c_n^-, n \in \mathbb{N}\} \subseteq S_\mu$.

(C2) *there exists* $\kappa_1 > \kappa_2 > 0$ *and* $\sigma_i : H \to \mathbb{R}, i \in \mathbb{N}$ *such that, for any* $i \in \mathbb{N}$,

- (D1) $\sigma_i : H \to \mathbb{R}$ *is continuous,*
- (D2) *for any* $h \in H$, $\kappa_2 < |\sigma_i(h)| < \kappa_1$.

(C3) *Let* $\beta_i \in \mathbb{R} \setminus \{0\}, i \in \mathbb{N}$ *be given constants such that*

$$\sum_{i=1}^{d} \int_{\mathbb{R}} |\beta_i x_i|^2 \wedge 1 \mu(dx_i) < \infty.$$

Example 5.7 continued

Denote by N_i the Poisson random measure corresponding to L_i, and $\tilde{N}_i$ the associated compensated Poisson random measure. Suppose that the noise term in (3.1) is of the form:

$$\begin{aligned}
& \int_0^t \int_{Z_1^c} \sigma(X(s-), z)\tilde{N}(dz, ds) + \int_0^t \int_{Z_1} \sigma(X(s-), z)N(dz, ds) \\
= & \sum_{i=1}^{d} \int_0^t \beta_i \sigma_i(X(s-))dL_i(s)e_i \\
= & \sum_{i=1}^{d} \Big(\int_0^t \int_{0<|z_i|\le 1} \beta_i \sigma_i(X(s-))z_i \tilde{N}_i(dz_i, ds)e_i + \int_0^t \int_{|z_i|>1} \beta_i \sigma_i(X(s-))z_i N_i(dz_i, ds)e_i \Big).
\end{aligned}$$

Then Assumption 3.3 holds.

Locally Monotone SPDEs

Recall that we consider the following SPDEs.

$$dX(t) = \mathcal{A}(X(t))dt + \int_{Z_1^c} \sigma(X(t-), z)\tilde{N}(dz, dt) + \int_{Z_1} \sigma(X(t-), z)N(dz, dt), \quad (5.4)$$
$$X(0) = x.$$

Suppose that there exist constants $\alpha > 1, \ \beta \geq 0, \ \theta > 0, \ C > 0, \ F > 0$ and a measurable (bounded on balls) function $\rho : V \to [0, +\infty)$ such that the following conditions hold for all $v, \ v_1, \ v_2 \in V$:

(H1) (Hemicontinuity) The map $s \mapsto_{V^*} \langle \mathcal{A}(v_1 + sv_2), v\rangle_V$ is continuous on $\mathbb{R}$,

(H2) (Local monotonicity)

$$\begin{aligned} & 2_{V^*}\langle \mathcal{A}(v_1) - \mathcal{A}(v_2), v_1 - v_2\rangle_V + \int_{Z_1^c} ||\sigma(v_1, z) - \sigma(v_2, z)||_H^2 \nu(dz) \\ \leq \quad & (C + \rho(v_2))||v_1 - v_2||_H^2, \end{aligned}$$

(H3) (Coercivity)

$$2_{V^*}\langle \mathcal{A}(v), v\rangle_V + \theta||v||_V^\alpha \leq F + C||v||_H^2,$$

(H4) (Growth)

$$||\mathcal{A}(v)||_{V^*}^{\frac{\alpha}{\alpha-1}} \leq (F + C||v||_V^\alpha)\big(1 + ||v||_H^\beta\big).$$

Now, we state the main result in this part.

Proposition 5.1

Under the conditions (H1)–(H4), assume that for any fixed $z \in Z_1$, $\sigma(\cdot, z) : H \to H$ *is continuous, and that the driving noise term satisfies Assumption 3.3, then the solution* $\{X^x, x \in H\}$ *to equation (5.4) is irreducible in* H.

Locally Monotone SPDEs

Our results in this subsection are applicable to SPDEs such as stochastic reaction–diffusion equations, stochastic semilinear evolution equation, stochastic porous medium equation, stochastic p-Laplace equation, stochastic Burgers type equations, stochastic 2D Navier-Stokes equation, stochastic magneto-hydrodynamic equations, stochastic Boussinesq model for the Bénard convection, stochastic 2D magnetic Bénard problem, stochastic 3D Leray-α model, stochastic equations of non-Newtonian fluids, several stochastic Shell Models of turbulence, and many other stochastic 2D Hydrodynamical systems.

Example 5.8 (p-Laplace equation)

Let Λ be an open bounded domain in $\mathbb{R}^d$, $d \in \mathbb{N}$, with smooth boundary. Consider the following Gelfand triple

$$V := W_0^{1,p}(\Lambda) \subseteq H := L^2(\Lambda) \subseteq (W_0^{1,p}(\Lambda))^*$$

and the stochastic p-Laplace equation

$$dX(t) = [\mathrm{div}(|\nabla X(t)|^{p-2}\nabla X(t))]dt + dL(t), \quad X(0) = x \in H, \tag{5.5}$$

where $p \in (1, \infty)$. We stress that this example covers the singular case, i.e., $p \in (1, 2)$.

Example 5.9

Let Λ *be an open bounded domain in* $\mathbb{R}^d$, $d \in \mathbb{N}$, *with smooth boundary. Consider the following Gelfand triple*

$$V := L^{r+1}(\Lambda) \subseteq H := W^{-1,2}(\Lambda) \subseteq (L^{r+1}(\Lambda))^*.$$

We consider the stochastic fast diffusion equation

$$\begin{aligned} & dX(t) = \Delta(|X(t)|^{r-1}X(t))dt + dL(t), \quad X(0) = x \in H, \quad (5.6) \\ & X(t,\xi) = 0, \ \forall \xi \in \partial\Lambda, \end{aligned}$$

where $r \in (0, \infty)$.

Nonlinear Schrödinger equations

Consider (3.3) with $H = L^2(\mathbb{R}^d)$, $d \in \mathbb{N}$, $\mathcal{A}(u) = i[\Delta u - \lambda |u|^{\alpha-1} u]$, where $\lambda \in \{-1, 1\}$, and $1 < \alpha < 1 + \frac{4}{d}$. Now consider NLS with additive noise, that is,

$$dX(t) = \mathcal{A}(X(t))dt + \int_{0<||z||_H\leq 1} z\tilde{N}(dz, dt) + \int_{||z||_H>1} zN(dz, dt), \qquad (5.7)$$
$$X(0) = x.$$

We say a pair (p, r) is admissible if $p, r \in [2, \infty]$ and $(p, r, d) \neq (2, \infty, 2)$ satisfying $\frac{2}{p} + \frac{d}{r} = \frac{d}{2}$.

The following result provides the existence and uniqueness of the solution of the stochastic NLS (5.7) whose proof was given in [WZ].

Proposition 5.2

Let $p \geq 2$, $1 < \alpha < 1 + \frac{4}{d}$, $r = \alpha + 1$ such that (p, r) is an admissible pair. For any $\hbar \in H$, there exists a unique global mild solution $X^\hbar = (X^\hbar(t), t \geq 0)$ of (5.7) satisfying

$$X^\hbar \in D([0, \infty); H) \cap L^p_{loc}(0, \infty; L^r(\mathbb{R}^d)), \ \mathbb{P}\text{-a.s.}$$

Nonlinear Schrödinger equations

Here is the result for the irreducibility of the solution.

Proposition 5.3

If the driving noise satisfies Assumption 3.4, then the solution $\{X^x, x \in H\}$ of (5.7) is irreducible in H.

Euler equations

We consider the stochastic Euler equation on $\mathbb{R}^2$:

$$d\mathbf{u}^\hbar(t) + (\mathbf{u}^\hbar(t)\cdot\nabla)\mathbf{u}^\hbar(t) = -\nabla p^\hbar(t) + \int_H zN(dz,dt), \qquad (5.8)$$
$$\mathrm{div}\ \mathbf{u}^\hbar(t) = 0, \quad \mathbf{u}^\hbar(0) = \hbar,$$

where $p^\hbar(t)$ is the scalar pressure. Let $k > 2$ be an integer and $H = H^k(\mathbb{R}^2)$.

Proposition 5.4

Let $\hbar \in H$. Then, there exists a unique solution $\mathbf{u}^\hbar \in D([0,\infty); H)$ of the equation (5.8), and $\{\mathbf{u}^\hbar, \hbar \in H\}$ forms a strong Markov process on H. If the driving noise satisfies Assumption 3.4, then $\{\mathbf{u}^\hbar, \hbar \in H\}$ is irreducible in H.

Singular SDEs

Let $L = (L_t)_{t\geq 0}$ be a Lévy process on $\mathbb{R}^d, d \in \mathbb{N}$, and denote its intensity measure by ν. To state the condition on ν, for $\alpha \in (0,2)$, denote by $\mathbb{L}^\alpha_{non}$ the space of all non-degenerate α-stable Lévy measure $\nu^{(\alpha)}$; that is,

$$\nu^{(\alpha)}(A) = \int_0^\infty \Big(\int_{\mathbb{S}^{d-1}} \frac{1_A(r\theta)\vartheta(d\theta)}{r^{1+\alpha}}\Big)dr, \quad A \in \mathcal{B}(\mathbb{R}^d),$$

where ϑ is a finite measure over the unit sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$ with

$$\inf_{\theta_0 \in \mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} |\theta_0 \cdot \theta| \vartheta(d\theta) > 0. \tag{5.9}$$

For $R > 0$, denote by B_R the closed ball in $\mathbb{R}^d$ centered at the origin with radius R. We assume that there are $\nu_1, \nu_2 \in \mathbb{L}^\alpha_{non}$, so that

$$\nu_1(A) \leq \nu(A) \leq \nu_2(A) \text{ for } A \in \mathcal{B}(B_1). \tag{5.10}$$

Singular SDEs

For a Borel measurable drift $b(\cdot) : \mathbb{R}^d \to \mathbb{R}^d$ and diffusion matrix $\sigma(\cdot) : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$, consider the following SDE

$$\begin{aligned} dX_t &= b(X_t)dt + \sigma(X_{t-})dL_t \\ &= b(X_t)dt + \int_{0<|z|\leq 1} \sigma(X_{t-})z\widetilde{N}(dz, dt) + \int_{|z|>1} \sigma(X_{t-})zN(dz, dt). \quad (5.11) \end{aligned}$$

Here N and $\widetilde{N}$ are the Poisson random measure and compensated Poisson random measure associated with L, respectively. In a recent paper [CZZ], the authors established the following well posedness of the SDE (5.11).

Lemma 5.1

Assume that ν satisfies (5.10) with $\alpha \in (0, 2)$. Assume that there are constants $\beta \in (1 - \alpha/2, 1]$ and $\Lambda > 0$ so that for all $x, y, \xi \in \mathbb{R}^d$,

$$|b(x)| \leq \Lambda \text{ and } |b(x) - b(y)| \leq \Lambda|x - y|^\beta, \quad (5.12)$$

$$\Lambda^{-1}|\xi| \leq |\sigma(x)\xi| \leq \Lambda|\xi| \text{ and } ||\sigma(x) - \sigma(y)|| \leq \Lambda|x - y|. \quad (5.13)$$

Then, there is a unique strong solution $X^x = (X^x(t), t \geq 0)$ to (5.11) for any initial data $x \in \mathbb{R}^d$.

We are concerned with the irreducibility of the solutions $\{X^x, x \in \mathbb{R}^d\}$ on $\mathbb{R}^d$. To obtain the irreducibility, we introduce the following conditions. Let $\{e_i\}_{i=1,2,\dots,d}$ be an orthonormal basis of $\mathbb{R}^d$.

(I) There exist $n \in \mathbb{N}$, $f_1, f_2, \dots, f_n \in \mathbb{S}^{d-1}$, and $\kappa \in (0,1]$, such that $\{f_1, f_2, \dots, f_n\} \subset S_\vartheta$, and for any $x \in \mathbb{R}^d$, $\inf_{y\in\mathbb{S}^{d-1}} \sup_{i=1,2,\dots,n} \frac{\langle \sigma(x) f_i, y\rangle}{|\sigma(x) f_i|} \geq \kappa$.

Proposition 5.5

Under the same assumptions of Lemma 5.1, and assume that (I) holds, the solutions to (5.11) is irreducible in $\mathbb{R}^d$.

Remark 5.3

The assumptions of Lemma 5.1 on the intensity measure ν of the Lévy process L are not sufficient to obtain the irreducibility. Here is one example: Let $d = 2$, choosing $x = (0,0)$, $b = (b_1, b_2), b_1, b_2 \geq 0$, $\vartheta = \delta_{e_1} + \delta_{e_2}$, here $e_1 = (1,0)$ and $e_1 = (0,1)$, $\sigma = I_{2\times 2}$ and

$$\nu(A) = \int_0^\infty (\int_{\mathbb{S}^{d-1}} \frac{1_A(r\theta)\vartheta(d\theta)}{r^{1+\alpha}})dr, \ \text{for } A \in \mathcal{B}(B_1), \ \nu(B_1^c) = 0.$$

Then $S_\nu = \{re_1, re_2, r \in (0,1]\}$, and $\{(c,d), c, d < 0\} \not\subset S_{Law(X_t^x)}$, $\forall t \geq 0$, and hence $\{X^x, x \in \mathbb{R}^d\}$ is not irreducible in $\mathbb{R}^d$.

Remark 5.3

The assumptions of Lemma 5.1 on the intensity measure ν *of the Lévy process* L *are not sufficient to obtain the irreducibility. Here is one example: Let* $d = 2$, *choosing* $x = (0,0)$, $b = (b_1, b_2), b_1, b_2 \geq 0$, $\vartheta = \delta_{e_1} + \delta_{e_2}$, *here* $e_1 = (1,0)$ *and* $e_1 = (0,1)$, $\sigma = I_{2\times 2}$ *and*

$$\nu(A) = \int_0^\infty \Big(\int_{\mathbb{S}^{d-1}} \frac{1_A(r\theta)\vartheta(d\theta)}{r^{1+\alpha}}\Big)dr, \ \text{for } A \in \mathcal{B}(B_1), \ \nu(B_1^c) = 0.$$

Then $S_\nu = \{re_1, re_2, r \in (0,1]\}$, *and* $\{(c,d), c, d < 0\} \not\subset S_{Law(X_t^x)}$, $\forall t \geq 0$, *and hence* $\{X^x, x \in \mathbb{R}^d\}$ *is not irreducible in* $\mathbb{R}^d$.

Our results in this subsection seem to be quite sharp. Because, on the one hand, the above example shows that if the condition (I) does not hold, then the solutions to (5.11) is not irreducible in $\mathbb{R}^d$. *On the other hand, the condition (I) seems a little stronger than (5.9). For example, if we replace* $\vartheta = \delta_{e_1} + \delta_{e_2}$ *in the above example by* $\vartheta = \delta_{e_1} + \delta_{e_2} + \delta_{e_3}$, *here* $e_3 = (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$, *then the condition (I) holds.*

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THANK YOU!